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Applications of Tensors and Differential Geometry in Relativity and Cosmology

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Abstract

In this paper we have discussed about the application of tensor calculus and differential geometry in the field relativity and cosmology. Tensor and differential geometry provide the mathematical foundation for relativity and cosmology, describing gravity as the curvature of space-time. In general relativity, the metric tensor defines space-time's geometric structure and is central part of the Field Equations given by Einstein, which link curvature to presence of mass and energy. This framework explains phenomena like gravitational lensing, black hole behaviour, and the expansion of the universe.

Keywords*:* Tensors, Differential Geometry, relativity, cosmology

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Introduction

In several fields of Mathematics, Physics and Engineering, there are two types of quantities. The first one is scalars and the second one is vectors. Scalars have only magnitude for examples mass; length, volume, density, work, etc are scalars. On the other hand, the vectors have magnitude as well direction. For example, velocity, acceleration force and momentum etc. are vectors. [Xu, Guangji and Hao Wang (2007), J. V, Narlikar (1978)].

Sometimes vectors are not sufficient to represent the physical quantities. In such case we need an extension of vectors which is called as tensor. A tensor is that quantities which include the information about the magnitude and direction of a physical quantity about that direction. For instance, the stress at a given point depends on two directions one perpendicular to the surface and the other which creates the stress therefore we can say that the stress cannot be represent by vector quantity. There are so many examples that led the extension of vectors quantities to higher quantity which is known as tensor [Woo *et al*. (2008), Z.Ahsan (1977, 1978, 1996, 1998, 2009, 2010)].

We can't assume the life in absence of tensors we cannot move in a room without any help of tensor which is called as tensor of pressure. It is somehow impossible to align the wheels of the car without the help of tensors. Definitely we cannot understand the Einstein theory of gravitation without the use of tensor [Corbett and Luke (1969), Yuan *et.al*. (2020), Hou *et al*. (2020)].

The basics concepts of tensors were given in DG (Differential Geometry) by G. Reimann and Christoffel in 19th century. The calculus on tensors was originated round 1887 by G. Ricci Curbastro and presented by Ricci in 1892. [Stephanin [1982], Szekeres (1968), Weinberg (1972), Yano (1940), Yano (1970), Yano and Kon (1984), Zakharov (1973), Pirani (1965), Sachs, (1964), Sharma and Husain (1969]. It was published in book form in 1901 in "Mathematische Annlen Vol. 54" by Ricci and Tullio Levi- Civita. In 20th century this subject became popular and was named as tensor analysis and became world famous when it was used in General theory of Relativity given by Albert Einstein in 1915 [Z. Ahsan (1977, 1978, 1996, 1998, 1999, 2009, 2010, 2013)].

By using Riemannian geometry and tensor analysis, Einstein gave concept of Gravitation through Einstein field equation in which space-time curve was described. He concluded that the mass considered as curvature in the space-time geometry. Using the techniques of Riemannian geometry, Einstein analysed interesting theory which predicts the nature of objects in the presence of gravitation field [J. Bergman, 2004, Ahsan and Hussain (1977)].

[A] Applications in different Co-ordinate Systems

2. Special Coordinate System

(i) Orthogonal Cordinate system

In generalised theory of relativity, we use the laws of physics using arbitrary coordinates. The principle of co-variance tells that the laws of physics are not cahnge in any coordinate system. In this part, we will analyse few of these choices. In vector calculus we use almost exclusively the familiar Cartesian coordinate system and other coordinate systems such as cylindrical and spherical coordinate system [A., Derdzinski, and C., Shen (1957, 1977)]. These three coordinate systems (Cartesian, cylindrical, spherical) are actually only a small subset of a larger group of coordinate systems which are orthogonal system. If in a coordinate system, the metric d is of the form

$$ds^2 = g^{ii} dx^i dx^i, \text{ i = 1,2,3,4}$$

i.e., the metric tensor has diagonal components only, then the coordinate system is said to be orthogonal coordinate system [Y. Ishii (1957), Keane and Tupper (2010), Kozameh et. al. (1985)]. In general, such coordinate system does not exist 4-D. Riemannian space. This is so because the system of differential equations

$$g'_{ij} \frac{dx'^i \, dx'^j}{dx^k dx^l} = 0 \text{ for } i \neq j$$

(ii)Time-orthogonal coordinate system

In physics, we usually talk about events and events occur at a certain place (represented by three space coordinates) and time [Sharma and Hussain (1969)]. So we choose time as the fourth coordinate and take:

$$x^4 = ct.$$

If $g_{4\alpha} = 0, (a = 1,2,3)$ then time orthogonal coordinates exist. Moreover, if $g_{44} = \pm 1$, wefind the Gaussian coordinate. In case of the time-orthogonal coordinates, the metric can express as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + g_{44} (dx^4)^2 \qquad (1).$$

3. Minkowski coordinate system

Minkowski coordinates are Cartesian space-time coordinates. In this co-ordinate system, the time component is multiplied by c to give in the dimensions of length which is speed of light [Sachs (1964), Pirani (1964)]. These coordinates are commonly written As

$$(x^1, x^2, x^3, x^4) = (x, y, z, ct). \qquad (2)$$

(i) Fermi coordinate system:

Those coordinate system which can be introduced to a geodesic are called Fermi coordinates [Yano (1970, 1984)]:

Let M be a n-dimensional Riemannian manifold, γ be a geodesic on M, let p be a point on γ.Then there exist local coordinates $(t, x^1, x^2, \ldots, x^n)$about p such that

(a) For small value of t, the coordinate (t, 0,..., 0) represents the geodesic around p,

(b) On γ, any metric tensor is the EM (Euclidean metric),

(c) All Christoffel's symbols do not exist on γ. This may be also noted that the Fermi coordinate system is valid only on the geodesic and thus, for example, if all Christoffel's symbols do not exist near 'p, then the geodesics are the straight lines and the manifold become flat around p.

(ii) Co-moving coordinate System:

Let $u'^i = \frac{dx'^i}{d\lambda}$be the velocity field (of observers). Since λ is a parameter which is independent of the coordinate, the component of this velocity transforms can be expressed as

$$u^j = \frac{dx^j}{dx'^k} u'^k \qquad (3)$$

This means there is possibility to select a coordinate system in which the spatial components of the velocity are zero, i.e., $u_1 = u_2 = u_3 = 0$. For the current coordinate system, the velocity is $u^i = (0,0,0,u^4)$, this implies that the particles do not change their positions this implies that the coordinates move with the time. This type of coordinate system is called co-moving co-ordinates. As the metric is time-dependent therefore the separation between two particles can change but the coordinate difference between them never changes. [Stephanin [1982], Szekeres (1968), Weinberg (1972), Yano (1940), Yano (1970), Yano and Kon (1984), Zakharov (1973), Pirani (1965), Sachs, (1964), Sharma and Hussain (1969].

The space is static in co-moving coordinates because most of the objects on the scale of galaxies are approximately co-moving and co-moving bodies have static, unchanging co-moving coordinates. Co-moving coordinates assign constant spatial coordinate values to observers who think that the universe looks like isotropic.Such observers are called co-moving observers because they are moving in a universe which is moving according to Hubble's law

Hubble's law tells that "the speed v of recession of a galaxy (an astronomical object) is directly proportional to its distance D from us", i.e. $r = HD$, where H is known as the Hubble's constant.

[B] Applications in Energy Momentum Tensors and Einstien Field Equation

2. Energy-momentum Tensor

Usually The energy-momentum tensors (also known as stress-energy tensors) occur in dynamics, the field hydrodynamics, electromagnetic theory and in the field theories in general. [Stephanin [1982], Szekeres (1968), Weinberg (1972), Yano (1940), Yano (1970), Yano and Kon (1984), Zakharov (1973), Pirani (1965), Sachs, (1964), Sharma and Husain (1969]. But their role is not so important in these areas it in general theory of relativity. When expressed in terms of the energy-momentum tensors which is the law of conservation for energy and momentum Minkowski take simpler form. In special theory of relativity which is also known as flat space-time, the value of energy E and linear momentum value p are the two aspects of a single quantity-the 4 momenta. These two are connected through the equation

$$E^2 - c^2p^2 = m^2c^4 \quad (4)$$

If ρ_0 is the proper density of the matter and $u^i = \frac{dx^i}{dt}$ denotes the motion of the matter then the tensor T^{ij} for energy momentum is defined as

$$T^{ij} = \rho_0 c^2 \frac{dx^i}{dt}\frac{dx^j}{dt} \quad (5)$$

If ρ is the coordinate density of matter, it can then easily be shown that energy momentum tensor defined above can takes the form

$$T^{ij} = \rho u^i u^j \quad (6)$$

The different components of T^{ij} have their meanings as follows:

(i) T^{44} is the energy density.

(ii) cT^4 is the energy flow per unit area parallel to the ith direction.

(iii) T^{ii} is the flow of momentum component i per unit area in the ith direction

(iv) T^{ij} is the flow of the ith componont of momentum per unit area in the jth direction.

(v) $T^{i4}\, c$ is the density of the ith component of momentum

The energy-momentum tensor T^{ij} also satisfies the equation

$$T^{ij}_{,j} = 0 \quad (7)$$

This shows that energy-momentum tensor is divergence-free everywhere or conserved. This result of special theory of relativity can be converted to a form of general theory of relativity (curved spacetime), if the partial derivatives are exchanged by co-variant derivatives. Thus, for the curved space-time equation (7) takes the form

$$T^{ij}_{,j} = 0 \quad (8)$$

which can also be expressed as

$$\frac{\partial T^{ij}}{\partial x^j} + \Gamma^j_{kj}T^{ik} + \Gamma^i_{kj}T^{kj} = 0 \quad (9)$$

It may be Moreover, noted that in the absence of material mass, the energy-momentum tensor is nothing. it is a second rank tensor whose divergence vanishes at all points. Einstein identified that energy-momentum tensor as the source of space time curvature and suggested the simple but important relation

$$R_{ij} - \frac{1}{2}Rg_{ij} = -KT_{ij} \quad (10)$$

Which is known as Einstein field equation in presence of matter [Stephanin [1982], Szekeres (1968), Weinberg (1972), Yano (1940), Yano (1970), Yano and Kon (1984), Zakharov (1973), Pirani (1965), Sachs, (1964), Sharma and Husain (1969].

Conclusion

Tensors and Differential geometry are indispensable for the general theory of relativity, providing mathematical structure for analysing the space-time curvature. This tensor provides a detailed account of how the geometry of the manifold changes from point to point. The concepts of manifolds, metrics, and curvature are essential for describing the gravitational interaction as a geometric phenomenon. Through the Einstein field equations, differential geometry connects the dispensation of energy and mass to the curvature of space-time, explaining a wide range of physical phenomena from black holes to the expansion of the universe [[Stephanin [1982], Szekeres (1968), Weinberg (1972), Yano (1940), Yano (1970), Yano and Kon (1984), Zakharov (1973), Pirani (1965), Sachs, (1964), Sharma and Husain (1969,]].

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